# A two-parameter accelerating FODO cell.

From circular reasoning to straight thinking.

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Although linacs are not storage rings, a similar language is used to describe their optical properties. This can be confusing. The periodicity of storage rings allows unambiguous definitions of "global lattice functions" – e.g., dispersion, beta, and phase advance – as properties of the machines themselves, with no reference to beam parameters other than energy. The "beta function" of a linac or the "phase advance" between two points in a linac cannot be defined in the same way. Further, in computing the lattice functions of a storage ring we assume a constant energy machine; this is extended to synchrotrons by assuming that the fields change adiabatically, over many turns. In contrast, acceleration can take place in a linac between consecutive quadrupoles.

We want to apply standard formulae from the theory of storage rings – which we will call "circular reasoning" (CR) – to a linac – whose theory we will call "straight thinking" (ST). These CR formulae involve lattice functions, and it is desirable to know how to apply them correctly and what sort of errors are made if they are applied without modification. To examine some of these questions, we will look at a simple model that straddles the two theories, a periodic structure that is a straightforward generalization of CR's FODO cell: the accelerating FODO cell. It will possess three formal parameters: the length of a drift section, the focal lengths of the quadrupoles, and a number representing the energy gain of the accelerating structure. However, the drift's length will serve to set the scale, so our model will have only two essential parameters.

## Transfer matrix of an acclerating FODO cell.

A simple accelerating FODO cell sketched in Figure 1. A drift section, not a bending magnet, separates the quads, and in it is placed a "thin" (zero length) accelerating structure. The quadrupoles are also thin, and their magnetic fields are arranged so that the focal length remains constant; that is, their gradients scale with momentum.

To first order, the effect of the accelerating structure on the transvere coordinates is to "rescale" the transverse momentum – or, equivalently, to reduce the transverse velocity – as represented by the transfer matrix,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{(f)} = \begin{pmatrix} 1 & 0 \\ 0 & p_i/p_f \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{(i)} .$$

Here $p_i$ and $p_f$ are the total momentum of the particle upon entering and exiting the

structure. We define $g \equiv p_i/p_f$ and write the transfer matrix for the full "drift" section.

$$\begin{aligned}
\mathbf{O} &= \begin{pmatrix} 1 & l/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & l/2 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & l(1+g)/2 \\ 0 & g \end{pmatrix}
\end{aligned}$$

In the limit of no acceleration, $g = 1$, while $g = 0$ represents extreme acceleration. For highly relativistic particles,

$$g = p_i/p_f \approx E_i/E_f = \frac{1}{1+(E_f - E_i)/E_i} = \frac{1}{1+\Delta E/E} \; .$$

The transfer matrices through the quads are the same as those used in CR, so we can immediately write the full transfer matrix through the accelerating FODO cell.

$$\begin{aligned}
\mathbf{M} &= \mathbf{F}\cdot\mathbf{O}\cdot\mathbf{D}\cdot\mathbf{O} \\
&= \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & l(1+g)/2 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & l(1+g)/2 \\ 0 & g \end{pmatrix} \\
&= \begin{pmatrix} 1 & l(1+g)/2 \\ -1/f & g - l(1+g)/2f \end{pmatrix} \begin{pmatrix} 1 & l(1+g)/2 \\ 1/f & g + l(1+g)/2f \end{pmatrix} \qquad (1) \\
&= \begin{pmatrix} 1 + (l/2f)(1+g) & (1+g)^2(l/2)(1+l/2f) \\ +(1/f)(g-1-(l/2f)(1+g)) & -(l/2f)(1+g) + (g^2 - ((l/2f)(1+g))^2) \end{pmatrix}
\end{aligned}$$

In writing all of this, we are finessing the question, "What is $x'$?" The variable that makes a drift a linear element is not the same as the one that makes a thin quadrupole a linear element. The former requires $x' = p_x/p_3 = dx/ds$ — where $p_x$ is the transverse and $p_3$ the longitudinal momentum, and $ds$ is longitudinal differential arc length — while the latter uses $x' = p_x/p$ — where $p$ is the total momentum of the particle, used as a reference scaling parameter. In confusing these two variables throughout the calculation, we implicitly require that $p \approx p_3$ and insist that the resulting equations are valid only to first order in the transverse variables. In this context, it is amusing to note that, at relativistic velocities, $dx/ds = (dx/dt)/(ds/dt)$ decreases because the accelerating structure actually *decelerates* the transverse component of velocity[1] while leaving the longitudinal component (essentially) unchanged.

**Eigenvalues.**

The eigenvalues of $\mathbf{M}$ determine the "phase advance" and amplitude reduction per cell. Because $\mathbf{M}$ is a matrix of real numbers, its eigenvalues come in complex conjugate pairs; because it is not symplectic, we do not expect the eigenvalues to lie on the unit circle. We express them as $\exp(-\Gamma \pm i\mu)$ and obtain them easily as follows.

$$\begin{aligned}
\det(\mathbf{M}) &= (e^{-\Gamma})^2 = g^2 \\
\mathrm{Tr}(\mathbf{M}) &= 2e^{-\Gamma}\cos\mu \\
&= 1 + g^2 - \left(\frac{l}{2f}(1+g)\right)^2
\end{aligned}$$

[1] That is, it decelerates in a direction orthogonal to the applied force.

The first equation is written quickly by noting that the determinant of a product is the product of the determinants and that $\det(\mathbf{O}) = g$, while $\det(\mathbf{F}) = \det(\mathbf{D}) = 1$. Rearranging terms slightly provides the results,

$$\begin{aligned} g &= e^{-\Gamma} && (2) \\ \left(\frac{l}{2f}\right)^2 &= \frac{1+g^2-2g\cos\mu}{(1+g)^2} \\ &= \frac{(1+g)^2-2g(1+\cos\mu)}{(1+g)^2} && (3) \\ &= 1-\frac{4g}{(1+g)^2}\cos^2(\mu/2)\ . \end{aligned}$$

In the limit $g = 1$ we obtain the usual relation,

$$\frac{l}{2f} = \sin(\mu/2)\ . \tag{4}$$

From the first line of Eq.(3), we see that the oscillatory condition, $|\cos\mu| \leq 1$, requires that the inequality

$$\frac{1-g}{1+g} \leq \frac{l}{2f} \leq 1$$

be satisfied. This is illustrated in Figure 2, which shows the surface generated by Eq.(3).

**<u>$\beta$, from the transfer matrix.</u>**

In CR, the eigenvectors of $\mathbf{M}$ are related to the "lattice functions" of the cell. In the limit $g = 1$, we are familiar with the relationship,

$$\frac{1}{\mu}\ln\mathbf{M}_{\mathrm{CR}} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\ . \tag{5}$$

This representation is possible because of the symplecticity of $\mathbf{M}_{\mathrm{CR}}$. Although the $\mathbf{M}$ of Eq.(1) is *not* symplectic, we can generalize Eq.(5) to it by factoring out the adiabatic damping. That is, $e^{\Gamma}\mathbf{M}$ *is* symplectic, so that we can write

$$\frac{1}{\mu}\ln\left(e^{\Gamma}\mathbf{M}\right) = \frac{1}{\mu}(\Gamma + \ln\mathbf{M}) = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \tag{6}$$

in place of Eq.(5). (Notice that $e^{\Gamma}\mathbf{M}$, $\mathbf{M}$, and $\ln\mathbf{M}$ all possess the same eigenvectors.) Because of Eq.(2) and Eq.(1), we can obtain the same result by altering the definition of $\mathbf{O}$.

$$\mathbf{O}_{symplectic} \equiv g^{-1/2}\mathbf{O} = \begin{pmatrix} g^{-1/2} & l(g^{-1/2}+g^{1/2})/2 \\ 0 & g^{1/2} \end{pmatrix}$$

Effectively, this rescales the transverse phase space coordinates to remove the amplitude reduction due to acceleration.

With this trick, we can calulate the "beta function" from the off-diagonal component of **M** in the usual circular reasoning manner.

$$\beta \sin\mu = e^{\Gamma} M_{12} = \frac{1}{g} M_{12}$$

Using Equations (1) and (3) and rearranging terms a little provide the result.

$$\beta/l = \frac{1}{2\sin\mu}\,(2+g+1/g)\left(1+\sqrt{1-\frac{4g}{(1+g)^2}\cos^2(\mu/2)}\right)$$

(This is $\beta$ at the location of the focussing quad; at the defocussing quad, merely replace $1+\sqrt{\cdots}$ with $1-\sqrt{\cdots}$.) A family of these curves is plotted in Figure 3 for discrete values of $g$ between 0.1 and 1. Unrealistically extreme acceleration would be required to observe a significant deviation from the $g = 1$ limit.

**Propagating lattice functions.**

We now compare expressions for "propagating" lattice functions in both CR and ST. In ST one defines "lattice functions" in terms of the covariance of the beam.

$$\mathbf{C} = \begin{pmatrix} \langle x^2\rangle & \langle xx'\rangle \\ \langle x'x\rangle & \langle x'^2\rangle \end{pmatrix} \equiv \det(\mathbf{C})^{1/2} \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \tag{7}$$

This is in agreement with Eq.(5) when the covariance matrix is that of an invariant ellipse of the one-turn matrix, $\mathbf{M}_{\mathrm{CR}}$. However, the beam need not be matched to the machine, so that $\alpha$ and $\beta$ are arbitrary here. Let $\mathbf{M}_{BA}$ symbolize the linear transfer matrix from point A to point B in the machine, so that

$$\mathbf{C}_{(B)} = \mathbf{M}_{BA} \cdot \mathbf{C}_{(A)} \cdot \mathbf{M}_{BA}^T \,.$$

Combining this with the definition in Eq.(7) provides the following.

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{(B)} = \frac{1}{\det(\mathbf{M}_{BA})}\,\mathbf{M}_{BA} \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{(A)} \cdot \mathbf{M}_{BA}^T \tag{8}$$

In formally eliminating **C** from this equation, we have used

$$\det(\mathbf{C}_{(B)}) = \det(\mathbf{C}_{(A)})\,\det(\mathbf{M}_{(BA)})^2 \,.$$

On the other hand, because of the connection with the period-advance map via Eq.(5),

$$\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}_{(B)} = \mathbf{M}_{BA} \cdot \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}_{(A)} \cdot \mathbf{M}_{BA}^{-1} \,. \tag{9}$$

This is valid even for $g < 1$ : according to Eq.(6), we simply subtract $\Gamma/\mu$ from both sides of the equation.

Superficially Eq.(8) and Eq.(9) look different, but one can be transformed into the other by using the $2\times 2$ matrix identity,

$$\begin{aligned}
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{M}^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} \\
&= \det(\mathbf{M})\, \mathbf{M}^{-1} \,.
\end{aligned}$$

We will use this with Eq.(8) to produce Eq.(9).

$$\begin{aligned}
&\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}_{(B)} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{(B)} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \frac{1}{\det(\mathbf{M}_{BA})}\, \mathbf{M}_{BA} \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{(A)} \cdot \mathbf{M}_{BA}^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \frac{1}{\det(\mathbf{M}_{BA})}\, \mathbf{M}_{BA} \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{(A)} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{M}_{BA}^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \mathbf{M}_{BA} \cdot \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}_{(A)} \cdot \mathbf{M}_{BA}^{-1}
\end{aligned}$$

Obviously, one can go in the other direction as well, producing Eq.(8) from Eq.(9), so that the two equations are indeed equivalent. It is therefore not remarkable that they both result in the expression:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_{(B)} = \begin{pmatrix} M_{11}M_{22} + M_{12}M_{21} & -M_{11}M_{21} & -M_{22}M_{12} \\ -2M_{11}M_{12} & M_{11}^2 & M_{12}^2 \\ -2M_{22}M_{21} & M_{21}^2 & M_{22}^2 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_{(A)}$$

**Error in phase advance.**

The accelerating FODO cell that we have described, while enabling us to calculate exact quantities, is not actually used as the basic unit of periodically structured linacs. In extending results to realistic linacs, one approach is to ignore acceleration except for its effect on scaling the transverse phase space variables. Thus, while $\beta$ actually depends on $g$, which changes from cell to cell, this can be safely ignored because the dependence is so weak for reasonable values of $\Delta E/E$. The phase advance per cell, $\mu$, also changes very little, but, unlike $\beta$, phase accumulates as one progresses down a chain of cells. The error that one would make by neglecting its dependence on $g$ can become significant after a sufficient number of cells. An estimate of this effect is shown in Figure 4. For a given $g$ and $\mu$, Eq.(3) is first used to find the parameter $l/2f$. Using this value, Eq.(4) then provides the (incorrect) phase advance, say $\mu_1$, obtained by ignoring acceleration. The difference, $\mu - \mu_1$, is plotted in Figure 4 as a function of $\mu$, the correct phase advance, while labelling each curve by the corresponding $\Delta E/E = 1/g - 1$. They indicate

that, for a $90^\circ$ phase advance per cell, and for $\Delta E/E$ even as large as 0.02, one would have to go through something like 100 cells before accumulating a $1^\circ$ phase error. This gives us an estimate of the extent over which we can trust standard CR formulae that involve the phase advance, provided we modify them to take phase space rescaling into account.

**Conclusions.**

For reasonable values of fractional energy gain per cell, the lattice functions of the accelerating FODO cell differ negligibly from that of the ordinary FODO cell. In particular, one must traverse hundreds of cells before accumulating an appreciable error in phase advance. The key step in connecting the CR and ST quantities is to remove the transverse phase space compression caused by the accelerating structure.

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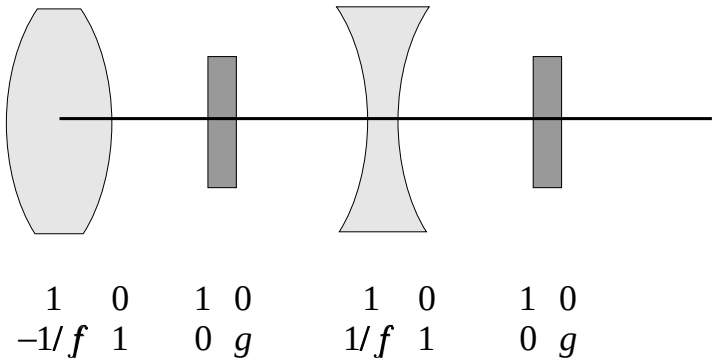


Figure 1: Sketch of an accelerating FODO cell.

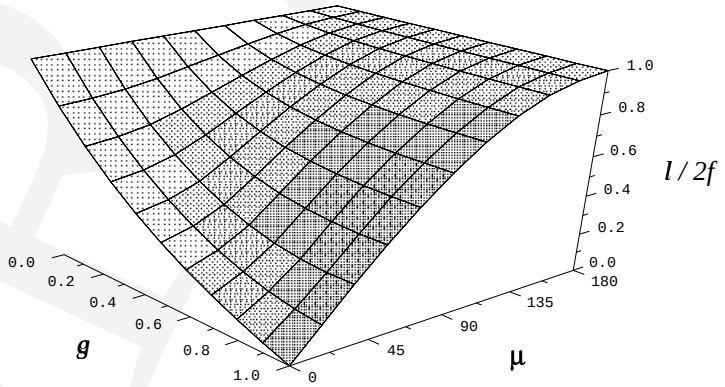


Figure 2: Ratio of quarter-cell length to focal length.

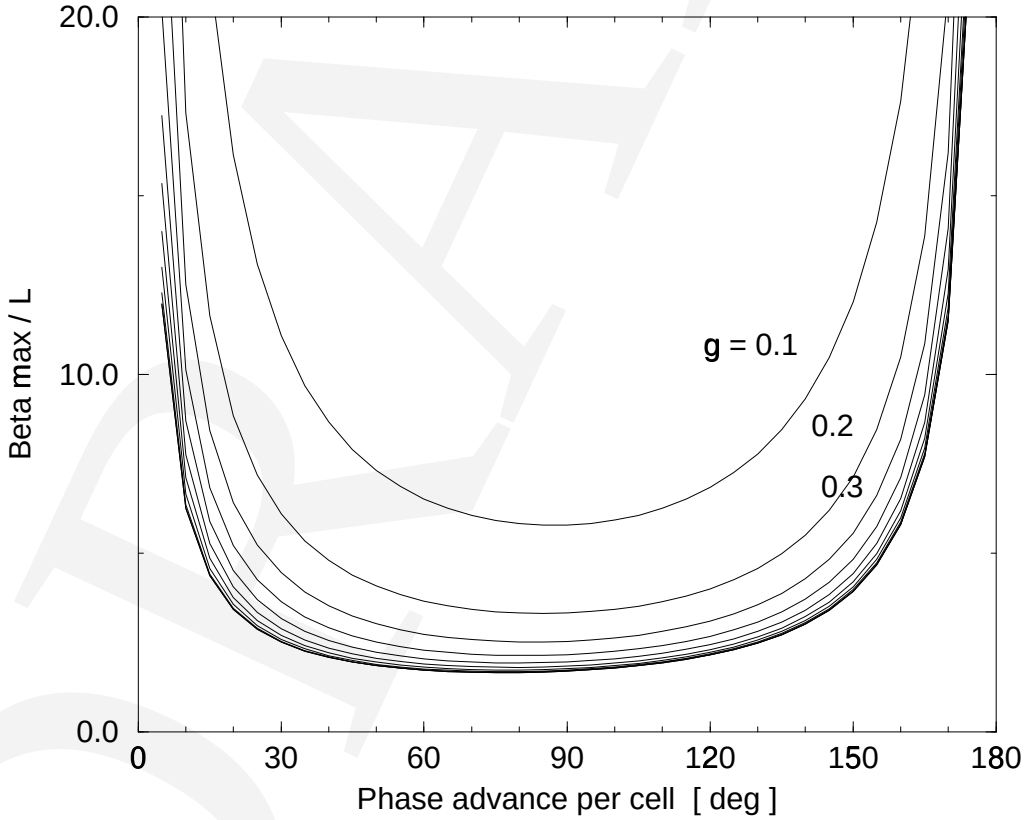


Figure 3: Maximum beta in the cell.

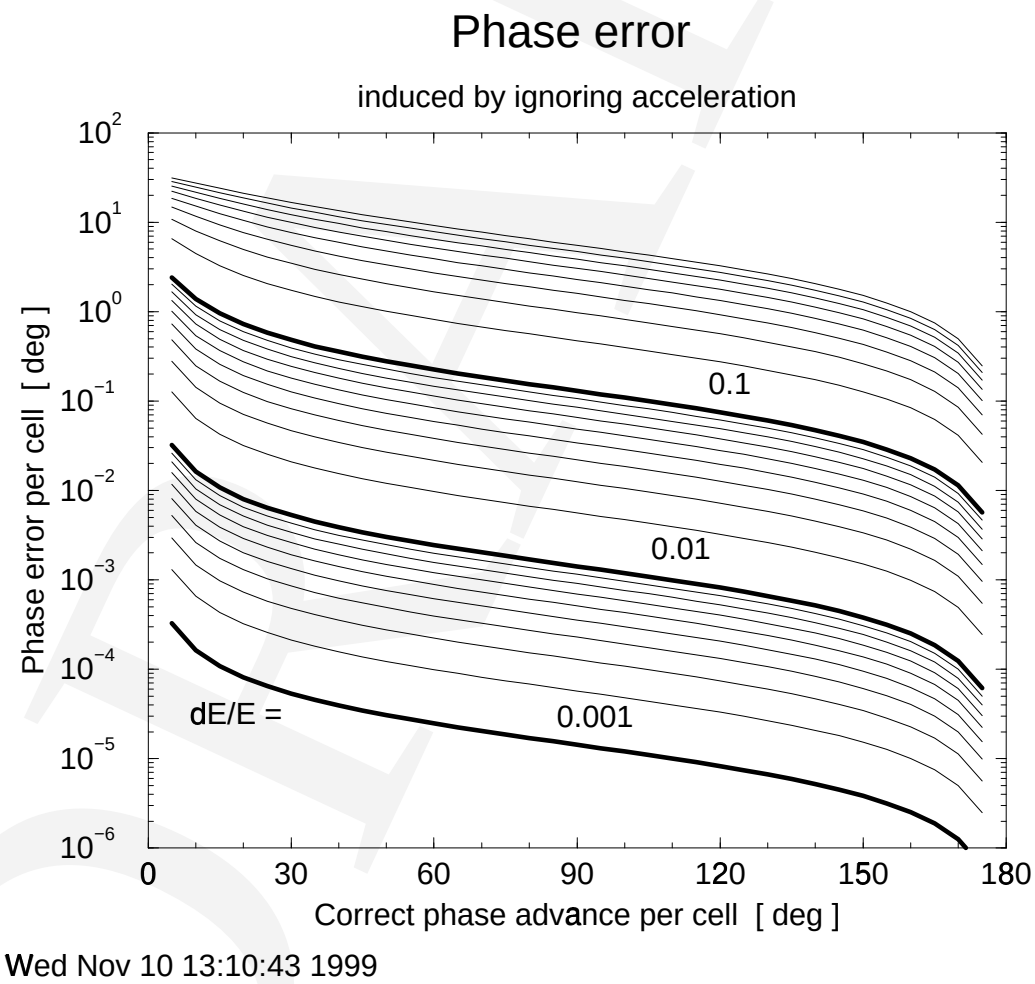


Figure 4: Phase error per cell induced by ignoring acceleration.